

A novel mechanism for exciting intrinsic toroidal rotation

C. J. McDevitt,^{1,a)} P. H. Diamond,¹ Ö. D. Gürçan,² and T. S. Hahm³

¹Department of Physics and Center for Astrophysics and Space Sciences,
University of California San Diego, La Jolla, California 92093-0424, USA

²CEA Cadarache, 13108 St. Paul Lez Durance, France

³Princeton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543-0451, USA

(Received 2 December 2008; accepted 31 March 2009; published online 4 May 2009)

Beginning from a phase space conserving gyrokinetic formulation, a systematic derivation of parallel momentum conservation uncovers two physically distinct mechanisms by which microturbulence may drive intrinsic rotation. The first mechanism, which emanates from $\mathbf{E} \times \mathbf{B}$ convection of parallel momentum, has already been analyzed [O. D. Gurçan *et al.*, Phys. Plasmas **14**, 042306 (2007); R. R. Dominguez and G. M. Staebler, Phys. Fluids B **5**, 3876 (1993)] and was shown to follow from radial electric field shear induced symmetry breaking of the spectrally averaged parallel wave number. Thus, this mechanism is most likely active in regions with steep pressure gradients or strong poloidal flow shear. The second mechanism uncovered, which appears in the gyrokinetic formulation through the parallel nonlinearity, emerges due to charge separation induced by the polarization drift. This novel means of driving intrinsic rotation, while nominally higher order in an expansion of the mode frequency divided by the ion cyclotron frequency, does not depend on radial electric field shear. Thus, while the magnitude of the former mechanism is strongly reduced in regions of weak radial electric field shear, this mechanism remains unabated and is thus likely relevant in complementary regimes. © 2009 American Institute of Physics.

[DOI: [10.1063/1.3122048](https://doi.org/10.1063/1.3122048)]

I. INTRODUCTION

Recent experimental studies have generated a substantial body of evidence suggesting that observations of intrinsic toroidal rotation encompass a rich, multifaceted set of phenomena. An understanding of the physical mechanisms driving intrinsic rotation is crucial to magnetic fusion since both toroidal flow shear and the rate of plasma rotation are well known to influence the power threshold required for L - H mode transitions as well as the stability of resistive wall modes. A crucial element of this diverse set of phenomena has been illuminated via the compilation of a broad database of H -mode plasma discharges.¹ This database suggests that within H -mode plasmas, the value of the offset of toroidal rotation typically scales linearly with the plasma stored energy divided by the plasma current and is usually in the cocurrent direction.²

While neoclassical, sub-neoclassical, and thermal ion orbit loss have been proposed as candidates for explaining various manifestations of intrinsic rotation,³⁻⁵ flow generation induced by turbulent stresses provides a natural candidate for many plasma regimes. Motivated by the latter of these two possibilities, a minimal model of intrinsic rotation has been constructed in Refs. 6 and 7 based on an L - H mode bifurcation model.⁸ This formulation, whose primary novel ingredient is a mean $\mathbf{E} \times \mathbf{B}$ shear driven contribution to the momentum flux (as well as convective terms, see Refs. 9-12, for example), suggests a close link between intrinsic rotation and regions of strong radial electric field shear, such as exist

near the edge of H -mode plasmas. Within the aforementioned framework, the rate of rotation in H -mode discharges is largely set by the width of the edge pedestal, which is typically closely correlated with the stored energy. Thus, this reduced framework appears to be capable of qualitatively reproducing several robust elements of the empirical trends observed in H -mode plasmas.

While certain empirical trends, especially those in H -mode plasmas, appear amenable to reduced models dependent on a single mechanism for driving rotation offsets, a significant subset of phenomena is *not* easily reconciled with the above trends. A particularly illustrative example is provided by recent experiments in L -mode limited plasma discharges.^{13,14} In these discharges, rates of plasma rotation in excess of that which can be accounted for by external momentum sources have been observed. Inversions in the direction of plasma rotation have been initiated via traversing a critical density (or current) threshold, which would seem to suggest either a mechanism whose sign inverts as the density threshold is crossed,¹⁵ or a competition between multiple independent mechanisms. Furthermore, the rotation velocity near the edge of the plasma is reduced by neutral drag, such that the role of scrape-off-layer flows is mitigated.¹⁶ Also, we note that these inversion events appear to be initiated in the plasma core, thus further distinguishing them from edge phenomena.

Before proceeding further it is useful to review the form of the turbulent momentum flux. The evolution of parallel momentum has been shown to obey¹⁷

^{a)}Electronic mail: cmcdevitt@ucsd.edu.

$$\frac{\partial \langle P_\phi \rangle}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(-\chi_\phi \frac{\partial \langle v_\phi \rangle}{\partial r} + V_c \langle v_\phi \rangle + S \right) \right] = S_{\text{ext}},$$

where for simplicity we do not distinguish the parallel and toroidal directions. The first term in the momentum flux corresponds to the toroidal viscosity, where early theoretical and experimental work have demonstrated the scaling $\chi_\phi \sim \chi_i$ (Refs 18 and 19) (although recent gyrokinetic simulations have shown significant departures from this scaling²⁰). The second term corresponds to a momentum pinch or convective term, and has been treated in Refs. 10–12 (see Refs. 7 and 21 for a discussion on the role of particle fluxes). Finally, the last term in the momentum flux is often referred to as the residual stress (i.e., the portion of the momentum flux independent of the mean toroidal velocity and its gradient), a particular manifestation of which has been shown to arise due to violations of reflectional symmetry about the rational surface, often induced by $\mathbf{E} \times \mathbf{B}$ shear.^{6,22–26} As discussed below, while the $\mathbf{E} \times \mathbf{B}$ shear induced residual stress term is not the only residual stress term present in the momentum flux, it is likely to be the most robust mechanism in the vicinity of transport barriers. This follows since while this portion of the residual stress is being enhanced by $\mathbf{E} \times \mathbf{B}$ shear, the remaining terms in the turbulent momentum flux are reduced due to $\mathbf{E} \times \mathbf{B}$ shear decorrelation of the background turbulence.^{27,28} Thus, the radial electric field shear driven portion of the residual stress is likely to play a conspicuous role in barrier regions. Assuming a stationary solution and integrating from 0 to r , this expression may be written as

$$\begin{aligned} & -\chi_\phi \frac{\partial \langle v_\phi(r) \rangle}{\partial r} + V_c \langle v_\phi(r) \rangle + S(r) \\ & = \frac{1}{r} \int_0^r dr' r' S_{\text{ext}}(r') = \hat{S}_{\text{ext}}(r), \end{aligned}$$

where \hat{S}_{ext} is the integrated external momentum source. Hence, while clearly physically distinct, the residual stress has a mathematical form which is isomorphic to the integrated momentum source. Thus, the residual stress corresponds to an ideal candidate for understanding offsets in the toroidal rotation velocity. This is in contrast to the convective or “pinch” term, whose magnitude is dependent on the local rotation velocity and thus has a fundamentally distinct effect on rotation profiles. Specifically, systems whose only nondiffusive contribution to the momentum flux is convective do not admit stationary solutions with a finite rate of rotation without the presence of either an external momentum source or a nonzero edge boundary condition. Thus, terms in the momentum flux with the form of a residual stress correspond to an ideal means for driving intrinsic rotation.

The diversity of experimental observations in L -mode plasmas^{14,29,30} naturally motivates the search for alternate mechanisms for driving intrinsic rotation. As discussed in the previous paragraph, residual stress terms correspond to an ideal candidate for understanding the spontaneous generation of parallel flows. Thus, the determination of the lowest order residual stress term is of vital importance. More precisely,

the modest level of mean $\mathbf{E} \times \mathbf{B}$ shear in L -mode plasmas potentially allows for alternate contributions to the residual stress term to enter, which are likely subdominant in H -mode plasmas. In this paper we present a careful analysis of the transport of parallel momentum induced by electrostatic microturbulence in a simplified geometry. Our motivation throughout this work is to investigate the role of ostensibly higher order nondiffusive terms in the momentum flux which have been neglected in existing theoretical analyses. In particular, the evolution of parallel momentum can be described via the first moment of the gyrokinetic equation, i.e.,

$$\begin{aligned} & \frac{\partial \langle P_\parallel \rangle}{\partial t} + \nabla_\perp \cdot \left\langle \sum_s m_s \int d^3 \bar{v} v_\parallel \dot{\mathbf{X}}_\perp \delta F_s \right\rangle \\ & = \left\langle \sum_s m_s \int d^3 \bar{v} \dot{V}_\parallel \delta F_s \right\rangle, \end{aligned}$$

where $\int d^3 \bar{v} \equiv 2\pi \int d\mu dv_\parallel B$, $\mu \equiv (1/2)v_\perp^2/B$, and the rest of the notation is standard. The second term on the left-hand side (LHS) of this expression can be shown to induce momentum transport via $\mathbf{E} \times \mathbf{B}$ convection of parallel momentum and is the origin of the aforementioned mean $\mathbf{E} \times \mathbf{B}$ shear driven momentum transport. In the electrostatic limit, the right-hand side (RHS) can be rewritten as

$$\begin{aligned} f_\parallel & = \left\langle \sum_s m_s \int d^3 \bar{v} \dot{V}_\parallel \delta F_s \right\rangle \\ & = - \left\langle \sum_s q_s \int d^3 \bar{v} \delta F_s \hat{b} \cdot \nabla J_0(k_\perp \rho_\perp) \delta \phi \right\rangle. \end{aligned}$$

Here we emphasize that δF_s represents the distribution of gyrocenters rather than particles. In order to understand the distinction between the gyrocenter distribution function and the particle distribution function, it is useful to consider the quasineutrality relation, i.e., $\delta n_e = \delta n_i$, where δn_e and δn_i represent the electron and ion perturbed particle densities. Rewriting the ion density perturbation in terms of an integral over the gyrocenter distribution function gives

$$\delta n_i \approx \int d^3 \bar{v} \left\{ J_0(k_\perp \rho_\perp) \delta F_i + F_i^{(0)} [J_0^2(k_\perp \rho_\perp) - 1] \frac{e \delta \phi}{T_i} \right\},$$

where the presence of the second term results from the polarization drift. Substituting this result back into the quasineutrality relation, rearranging terms, allows the quasineutrality condition to be written in the suggestive form^{31,32}

$$\epsilon_\perp \nabla_\perp^2 \delta \phi = -4\pi \sum_s q_s \int d^3 \bar{v} J_0(k_\perp \rho_\perp) \delta F_s,$$

where $\epsilon_\perp \approx c^2/v_A^2 = \rho_s^2/\lambda_{De}^2$, and we note that for ions $\rho_\perp \equiv |v_\perp|/\omega_{ci}$, whereas for electrons, we take the limit $m_e \rightarrow 0$, such that $J_0 = 1$. Thus, it is clear that even in the limit of vanishing Debye length, gyrocenter quasineutrality is not satisfied. The violation of gyrocenter quasineutrality is well known to follow from the polarization drift appearing in the theory as an effective polarization shielding in the gyrokinetic Poisson equation, rather than explicitly appearing in the

gyrocenter equations of motion. Similarly, for $(k_{\perp}\rho_s)^2 \approx 1$ it is easy to see that the LHS and RHS of the gyrokinetic Poisson equation contribute at the same order.

The general form of the gyrokinetic Poisson equation can be easily seen to be closely analogous to the Poisson equation for a dielectric medium.³³ In order to demonstrate this more explicitly, it is useful to write the latter in the form

$$\epsilon \nabla_{\perp}^2 \delta\phi = -4\pi \sum_s q_s \int d^3v \delta f_s.$$

Thus, while not precisely isomorphic to the gyrokinetic Poisson equation, this expression has a very similar structure. This observation naturally raises the question as to the fate of the gyrokinetic analog of the parallel component of the Maxwell stress tensor, i.e., for a dielectric medium the parallel force density is given in terms of the electrostatic field as $f_{\parallel} = \partial \sigma_{x,\parallel} / \partial x$, where $\sigma_{x,\parallel} \equiv \epsilon / (4\pi) E_x E_{\parallel}$,³⁴ x is a radial variable, and we are only considering radial fluxes. In the following, the gyrokinetic analog of the Maxwell stress tensor for an electrostatic plasma is derived. It is shown that the momentum flux induced by this mechanism is independent of both the mean toroidal velocity and its gradient, and hence can be classified as a residual stress term. Terms of this form have been shown to have an ideal mathematical form for driving intrinsic rotation.

This mechanism, while naively appearing to be higher order in $\epsilon \sim \rho_i / L_n$, is not tied to radial electric field shear in contrast to its more familiar mean $\mathbf{E} \times \mathbf{B}$ shear driven counterpart. More specifically, the mean $\mathbf{E} \times \mathbf{B}$ shear driven component of the residual stress can be shown to be proportional to the spectrally averaged parallel wave number $\langle k_{\parallel} / k_y \rangle$.⁶ This quantity in turn is proportional to the shift in the radial eigenmode off its rational surface, which typically scales as

$$\langle k_{\parallel} / k_y \rangle \sim x_0 / L_s \approx A_s \omega_{ci}^{-1} v'_E,$$

where v'_E is the mean $\mathbf{E} \times \mathbf{B}$ flow shear and A_s is a mode dependent parameter whose magnitude based on simple models typically satisfies $L_n / L_s < |A_s| < L_s / L_n$ (see Appendix B). If we estimate v'_E via the diamagnetic term in the radial force balance equation, this yields $v'_E \approx v_{thi} \rho_i (1 + \eta_i) / L_n^2$, so we may estimate the spectrally averaged k_{\parallel} / k_y by

$$\langle k_{\parallel} / k_y \rangle \approx A_s (\rho_i / L_n)^2 (1 + \eta_i).$$

Hence, while a naive ordering would suggest $k_{\parallel} / k_y \sim \epsilon$, it is clear that after summation over the turbulence spectrum $\langle k_{\parallel} / k_y \rangle \sim \epsilon^2$ (i.e., one order higher than would be naively anticipated). This simple analysis suggests that the contribution to the residual stress emanating from $\mathbf{E} \times \mathbf{B}$ convection can enter at the same order as contributions due to the polarization drift. Thus, the self-consistent determination of the turbulent residual stress term, which is likely necessary for describing elements of the aforementioned phenomena, requires the inclusion of contributions to the turbulent momentum flux emanating from the polarization drift.

The remainder of this paper is organized as follows. In Sec. II a systematic derivation of the parallel momentum conservation theorem from a phase space conserving gyrokinetic equation is carried out to fourth order in $\epsilon \sim \rho_i / L_n$.

Section III presents a discussion of momentum transport driven by both $\mathbf{E} \times \mathbf{B}$ flow shear, as well as a novel mechanism which arises from the parallel nonlinearity within the gyrokinetic framework. Comparisons of these two mechanisms are performed with an eye toward understanding in which regime each mechanism is likely to dominate. Section IV presents a brief summary.

II. MOMENTUM CONSERVATION THEOREM

In this section we present a derivation of a parallel momentum conservation theorem beginning from a phase space conserving gyrokinetic equation. It will be convenient throughout this analysis to utilize a multiscale perturbative framework, which provides a systematic means of identifying diverse contributions to the momentum evolution equation, as well as providing a transparent mechanism for ordering each contribution's apparent magnitude. Our emphasis here will be on determining the lowest order contributions emanating from $\mathbf{E} \times \mathbf{B}$ convection, as well as the polarization drift.

The electrostatic gyrokinetic equation can be written as

$$0 = \frac{\partial F_s}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\dot{\mathbf{X}} F_s) + \frac{\partial}{\partial v_{\parallel}} (\dot{V}_{\parallel} F_s), \quad (1a)$$

where

$$\dot{\mathbf{X}} = \hat{b} v_{\parallel} + \frac{c}{B} \hat{b} \times \nabla (\langle \delta\phi \rangle_{\alpha} + \Phi), \quad (1b)$$

$$\dot{V}_{\parallel} = -\frac{q_s}{m_s} \hat{b} \cdot \nabla \langle \delta\phi \rangle_{\alpha}. \quad (1c)$$

Here, the subscript s represents the species of particle, $\langle \dots \rangle_{\alpha} \equiv (2\pi)^{-1} \int_0^{2\pi} d\alpha (\dots)$, Φ is the equilibrium scalar potential (assumed to be only a function of the radial variable x), and for simplicity, we consider cylindrical geometry.

In order to derive a general expression for the evolution of parallel momentum it is useful to separate the temporal and perpendicular spatial scales into a set of "fast" variables associated with the rapidly varying microturbulence, which we will denote by (\mathbf{x}_{\perp}, t) , and a set of "slow" variables typical of equilibrium profiles denoted by (\mathbf{X}_{\perp}, T) (see, for example, Ref. 35). This separation allows for the decomposition of the perpendicular space and time derivatives in the form

$$\nabla_{\perp} \rightarrow \nabla_{\perp}^{(0)} + \epsilon \nabla_{\perp}^{(1)}, \quad \frac{\partial}{\partial t} \rightarrow \epsilon \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T}. \quad (2)$$

Here $\nabla_{\perp}^{(0)}$ corresponds to a derivative with respect to \mathbf{x}_{\perp} and $\nabla_{\perp}^{(1)}$ corresponds to a derivative with respect to \mathbf{X}_{\perp} . We note that while the fast and slow variables should be regarded as independent, the fast and slow derivatives do not necessarily commute with one another. For example, in cylindrical geometry it is easy to see that $\nabla_{\perp}^{(1)} \cdot \nabla_{\perp}^{(0)} \neq \nabla_{\perp}^{(0)} \cdot \nabla_{\perp}^{(1)}$. The parallel derivative is ordered as

$$\hat{b} \cdot \nabla \sim \varepsilon \hat{b} \cdot \nabla, \quad (3)$$

where $\varepsilon \sim \rho_i/L_n$. Note that since functions of only the large scales are assumed to be uniform along magnetic field lines, there is no need to introduce an analogous decomposition in the parallel direction. While for the simple geometry utilized here, an analogous statement can be made for the poloidal variable as well, it is convenient to introduce this scale separation such that the expressions derived below have a greater range of applicability and a more transparent form. If we now assume the fluctuating fields to be approximately described by their mixing length levels, we can introduce the ordering

$$\delta\phi = \varepsilon \delta\phi^{(1)}(\mathbf{x}, t, \mathbf{X}_\perp, T) + \varepsilon^2 \delta\phi^{(2)}(\mathbf{x}, t, \mathbf{X}_\perp, T) + \dots,$$

$$F_s = F_s^{(0)}(\mathbf{X}_\perp) + \varepsilon \delta F_s^{(1)}(\mathbf{x}, t, \mathbf{X}_\perp, T) + \varepsilon^2 \delta F_s^{(2)}(\mathbf{x}, t, \mathbf{X}_\perp, T) + \dots,$$

where $F_s^{(0)}$ is taken to be a centered Maxwellian. Furthermore, we may define a spatial and temporal average over the fast scales such that $\langle \delta\psi(\mathbf{x}, t, \mathbf{X}_\perp, T) \rangle = 0$, but functions of only slow variables are left unaltered, i.e., $\langle \psi(\mathbf{X}_\perp, T) \rangle = \psi(\mathbf{X}_\perp, T)$. Similarly, averages over the fast scales annihilate derivatives of fast variables as well as derivatives along magnetic field lines but commute with slow derivatives, i.e., $\langle \nabla_\perp^{(0)} \psi \rangle = \langle \hat{b} \cdot \nabla \psi \rangle = 0$, but $\langle \nabla_\perp^{(1)} \psi \rangle = \nabla_\perp^{(1)} \langle \psi \rangle$.

Here, it is useful to derive properties of the $J_0(k_\perp \rho_\perp)$ operator since this operator will appear frequently in the ensuing analysis. Writing this operator in terms of the fast and slow variables introduced above, we find

$$J_0(\lambda) \approx 1 + \frac{1}{4} \rho_\perp^2 [(\nabla_\perp^{(0)})^2 + \varepsilon (\nabla_\perp^{(1)} \cdot \nabla_\perp^{(0)} + \nabla_\perp^{(0)} \cdot \nabla_\perp^{(1)}) + \varepsilon^2 (\nabla_\perp^{(1)})^2] + \dots,$$

such that we may define

$$J_0^{(0)}(\lambda) \equiv 1 + (1/4) \rho_\perp^2 (\nabla_\perp^{(0)})^2 + \dots,$$

$$J_0^{(1)}(\lambda) \equiv \frac{1}{4} \rho_\perp^2 (\nabla_\perp^{(1)} \cdot \nabla_\perp^{(0)} + \nabla_\perp^{(0)} \cdot \nabla_\perp^{(1)}) + \dots,$$

where $\lambda \equiv k_\perp \rho_\perp$ for ions $\rho_\perp \equiv |\mathbf{v}_\perp|/\omega_{ci}$ and $\omega_{ci} \equiv eB/(m_i c)$. Hence, while $J_0^{(0)}$ commutes with fluctuation quantities inside averages (i.e., it involves an even number of integrations by parts and the surface terms vanish), $J_0^{(1)}$ cannot be commuted without introducing surface terms.

A general expression for the evolution of parallel momentum can be obtained by operating on Eq. (1) with $\sum_s m_s \int d^3 \bar{v} v_\parallel$ and averaging over the fast scales, which yields

$$\begin{aligned} & \left\langle \frac{\partial P_\parallel}{\partial t} \right\rangle + \left\langle \nabla \cdot \sum_s m_s \int d^3 \bar{v} \dot{X}_\parallel F_s \right\rangle \\ &= \left\langle \sum_s m_s \int d^3 \bar{v} \dot{V}_\parallel F_s \right\rangle, \end{aligned} \quad (4)$$

where $P_\parallel \equiv \sum_s m_s \int d^3 \bar{v} v_\parallel F_s$, $\int d^3 \bar{v} \equiv 2\pi \int d\mu dv_\parallel B$, and $\mu \equiv v_\perp^2/(2B)$. Equation (4) may be simplified via an expansion in ε . Considering the first term on the LHS of Eq. (4), this term can be simplified to

$$\begin{aligned} \left\langle \frac{\partial P_\parallel}{\partial t} \right\rangle &= \left\langle \left(\varepsilon \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T} \right) \sum_s m_s \int d^3 \bar{v} v_\parallel [F^{(0)}(\mathbf{X}_\perp) \right. \\ &\quad \left. + \varepsilon \delta F_s^{(1)}(\mathbf{x}, t, \mathbf{X}_\perp, T) + \dots \right] \right\rangle \\ &\approx \varepsilon^3 \frac{\partial}{\partial T} \left\langle \sum_s m_s \int d^3 \bar{v} v_\parallel \delta F^{(1)}(\mathbf{x}, t, \mathbf{X}_\perp, T) \right\rangle \\ &= \varepsilon^3 \frac{\partial}{\partial T} \langle \delta P_\parallel^{(1)} \rangle. \end{aligned} \quad (5)$$

Formally $\delta F_s^{(1)}$ vanishes upon averaging; however the momentum theorems derived below will be more transparent with the inclusion of this term. Similarly, the second term on the LHS of Eq. (4) may be written to lowest order as

$$\begin{aligned} & \left\langle \nabla \cdot \sum_s m_s \int d^3 \bar{v} v_\parallel \dot{X}_\parallel F_s \right\rangle \\ &\approx \varepsilon^3 \frac{c}{B} \nabla_\perp^{(1)} \cdot \left\langle \sum_s m_s \int d^3 \bar{v} v_\parallel \delta F_s^{(1)} \right. \\ &\quad \left. \times (\hat{b} \times \nabla_\perp^{(0)} J_0^{(0)}(\lambda) \delta\phi^{(1)}) \right\rangle, \end{aligned} \quad (6)$$

such that the lowest order surviving term enters at $O(\varepsilon^3)$ and can be recognized as describing momentum transported by $\mathbf{E} \times \mathbf{B}$ convection.

Turning now to the RHS of Eq. (4), in the drift kinetic limit [i.e., $(k_\perp \rho_i)^2 \rightarrow 0$] this term trivially vanishes due to quasineutrality. However, for the more general limit of $(k_\perp \rho_i)^2 \approx 1$, the RHS of Eq. (4) is in general nonvanishing due to the violation of gyrocenter quasineutrality. This can be transparently demonstrated by writing the gyrokinetic Poisson equation in the form^{31,36}

$$\frac{\varepsilon_\perp}{4\pi n_0} \nabla_\perp \cdot (n_i \nabla_\perp \delta\phi) = - \sum_s q_s \int d^3 \bar{v} J_0(\lambda) \delta F_s, \quad (7)$$

where $\varepsilon_\perp \approx c^2/v_A^2 \gg 1$,³³ we have taken $\lambda_{De} \rightarrow 0$, and for simplicity we assume $k_\perp \rho_i < 1$ so that terms of order $O(k_\perp^4 \rho_i^4)$ may be neglected. Deviations from gyrocenter quasineutrality are well known to result from the presence of the polarization drift [absent from Eq. (6)]. Hence, the evaluation of this term is required in order to compute the contribution to the momentum flux originating from the polarization drift.

To second order, the RHS of Eq. (4) can be written as

$$f_\parallel^{(2)} = \left\langle \sum_s m_s \int d^3 \bar{v} \dot{V}_\parallel^{(2)} F_s^{(0)} \right\rangle, \quad (8)$$

where

$$\dot{V}_\parallel^{(2)} = - \frac{q_s}{m_s} J_0^{(0)}(\lambda) \hat{b} \cdot \nabla \delta\phi^{(1)}. \quad (9)$$

Substituting Eq. (9) into Eq. (8) yields

$$f_{\parallel}^{(2)} = - \left\langle \sum_s q_s \int d^3\bar{v} F_s^{(0)} J_0^{(0)}(\lambda) \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle = 0,$$

such that consistent with the previous two terms, the parallel force vanishes at second order. The third order parallel force may be written as

$$f_{\parallel}^{(3)} = \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(2)} \delta F_s^{(1)} \right\rangle + \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(3)} F_s^{(0)} \right\rangle, \quad (10)$$

where

$$\dot{V}_{\parallel}^{(3)} = - \frac{q_s}{m_s} \hat{b} \cdot \nabla \{ J_0^{(0)}(\lambda) \delta\phi^{(2)} + J_0^{(1)}(\lambda) \delta\phi^{(1)} \}, \quad (11)$$

and the gyrokinetic Poisson equation is given to first order by

$$\epsilon_{\perp} (\nabla_{\perp}^{(0)})^2 \delta\phi^{(1)} = -4\pi \sum_s q_s \int d^3\bar{v} J_0^{(0)}(\lambda) \delta F_s^{(1)}. \quad (12)$$

Considering the second term in Eq. (10) first, this term can be rewritten after substitution of Eq. (11) as

$$\left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(3)} F_s^{(0)} \right\rangle = 0. \quad (13)$$

Turning now to the first term in Eq. (10), after utilizing Eq. (9) this term can be written in the form

$$\left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(2)} \delta F_s^{(1)} \right\rangle = - \left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} J_0^{(0)}(\lambda) \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle, \quad (14)$$

which may be simplified via substitution of the first order Poisson equation, i.e.,

$$\begin{aligned} & - \left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} J_0^{(0)}(\lambda) \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle \\ &= \frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(0)})^2 \delta\phi^{(1)} (\hat{b} \cdot \nabla) \delta\phi^{(1)} \rangle \\ &= - \frac{\epsilon_{\perp}}{4\pi} \langle \nabla_{\perp}^{(0)} \delta\phi^{(1)} \cdot (\hat{b} \cdot \nabla) \nabla_{\perp}^{(0)} \delta\phi^{(1)} \rangle \\ &= - \frac{\epsilon_{\perp}}{8\pi} \langle \hat{b} \cdot \nabla | \nabla_{\perp}^{(0)} \delta\phi^{(1)}|^2 \rangle = 0, \end{aligned}$$

such that the diagonal component of the gyrokinetic analog of the Maxwell stress tensor can be seen to vanish upon averaging. Continuing to fourth order, $f_{\parallel}^{(4)}$ can be written as

$$f_{\parallel}^{(4)} = \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(2)} \delta F_s^{(2)} \right\rangle + \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(3)} \delta F_s^{(1)} \right\rangle + \left\langle \sum_s m_s \int d^3\bar{v} \dot{V}_{\parallel}^{(4)} F_s^{(0)} \right\rangle, \quad (15)$$

where

$$\dot{V}_{\parallel}^{(4)} = - \frac{q_s}{m_s} [J_0(\lambda) \hat{b} \cdot \nabla \delta\phi]^{(4)}, \quad (16)$$

with the second order Poisson equation defined as

$$\begin{aligned} & \epsilon_{\perp} (\nabla_{\perp}^{(0)})^2 \delta\phi^{(2)} + \frac{\epsilon_{\perp}}{n_0} \nabla_{\perp}^{(1)} \cdot (n_0 \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \\ &+ \epsilon_{\perp} \nabla_{\perp}^{(1)} \cdot \nabla_{\perp}^{(0)} \delta\phi^{(1)} + \frac{\epsilon_{\perp}}{n_0} \nabla_{\perp}^{(0)} \cdot (\delta n_i^{(1)} \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \\ &= -4\pi \sum_s q_s \int d^3\bar{v} [J_0^{(0)}(\lambda) \delta F_s^{(2)} + J_0^{(1)}(\lambda) \delta F_s^{(1)}]. \quad (17) \end{aligned}$$

$f_{\parallel}^{(4)}$ can be simplified by following a similar analysis as for the third order term such that Eq. (15) can be reduced to (see Appendix A for details)

$$f_{\parallel}^{(4)} \approx \frac{\epsilon_{\perp}}{4\pi n_0} \nabla_{\perp}^{(1)} \cdot \langle n_0 \delta \mathbf{E}_{\perp}^{(1)} \delta E_{\parallel}^{(2)} \rangle - \frac{1}{2} \nabla_{\perp}^{(1)} \cdot \left\langle \sum_s q_s \int d^3\bar{v} \rho_{\perp}^2 (\nabla_{\perp}^{(0)} \delta F_s^{(1)}) \delta E_{\parallel}^{(2)} \right\rangle, \quad (18)$$

where $\delta \mathbf{E} \equiv -\nabla \delta\phi$ and we have made the approximation $J_0^{(1)}(\lambda) \approx (1/4) \rho_{\perp}^2 (\nabla_{\perp}^{(1)} \cdot \nabla_{\perp}^{(0)} + \nabla_{\perp}^{(0)} \cdot \nabla_{\perp}^{(1)})$. The coefficient in front of the first term in Eq. (18) can be rewritten as $\epsilon_{\perp} / (4\pi n_0) \approx (c/v_A)^2 / (4\pi n_0) = m_i (c/B)^2$, which for the simplified geometry employed is a constant. Thus, $f_{\parallel}^{(4)}$ can be easily seen to describe a turbulent flux of momentum. Equation (18) provides the relevant generalization of the electrostatic Maxwell stress tensor to the gyrokinetic framework. While the first term can be identified as directly analogous to an off-diagonal component of the Maxwell stress tensor, the origin of the second term is somewhat less clear. This term originates from the Bessel function dependence in Eqs. (7) and (9) and will be shown below to enforce the quasistatic limit of the polarization drift. Note that while $f_{\parallel}^{(4)}$ naively appears to enter one order higher in $\epsilon \sim \rho_i / L_n$ than Eq. (6) as shown below, the residual stress term emanating from the momentum flux described by Eq. (6) vanishes to the lowest order, requiring consideration of both of these terms.

III. MOMENTUM FLUX

In this section we compute the nondiffusive components of the momentum flux induced both by $\mathbf{E} \times \mathbf{B}$ convection and the polarization drift in the electrostatic limit. The first of these two mechanisms has already been extensively studied in the existing literature (see, for example, Refs. 6, 22–25,

and 37–39) and we will thus limit ourselves to a brief review with an emphasis on estimating when this term is likely to dominate the polarization drift term.

A. $\mathbf{E} \times \mathbf{B}$ convection

We begin by rewriting Eq. (6) in the form

$$\Pi_{\parallel}^{EB} = i \frac{c}{B} \left\langle \sum_s m_s \sum_{m,n} \frac{m}{r} \int d^3 \bar{v} v_{\parallel} \delta F_{m,n}^{(s)} J_0(\lambda) \delta \phi_{-m,-n} \right\rangle_x, \quad (19)$$

where we have neglected ordering superscripts for simplicity, defined $\langle \cdots \rangle_x \equiv \int_{-\infty}^{\infty} dx (\cdots)$, $x = r - r_{m,n}$, and introduced the Fourier transform $[\delta \phi(\mathbf{x}, t), \delta F_s(\mathbf{x}, t)] = \sum_{m,n} [\delta \phi_{m,n}(x), \delta F_{m,n}^{(s)}(x)] \exp[i(m\theta - nz/R - \omega_k t)]$. An explicit quasilinear expression for the momentum flux in terms of $\delta \phi_{m,n}$ can be obtained via linearizing Eq. (1), yielding

$$\delta F_{m,n}^{(s)}(x) = -g_{m,n} \left\{ \frac{c}{B} k_y \left(\frac{\partial F_s^{(0)}}{\partial x} - \frac{\partial v_{\parallel}^{(0)}}{\partial x} \frac{\partial F_s^{(0)}}{\partial v_{\parallel}} \right) - \frac{q_s}{T_s} v_{\parallel} k_{\parallel} F_s^{(0)} \right\} J_0(\lambda) \delta \phi_{m,n}(x), \quad (20a)$$

where the response function is given by

$$g_{m,n} \equiv (\omega_k - v_{\parallel} k_{\parallel} - v_E^{(0)} k_y)^{-1}. \quad (20b)$$

Here, we have defined $k_y \equiv -i(1/r)(\partial/\partial\theta)(m\theta - nz/R) = m/r$, $k_{\parallel} \equiv -i\hat{b} \cdot \nabla(m\theta - nz/R) = (B_{\theta}/Br)(m - nq)$, $v_E^{(0)} \equiv (c/B) \partial\Phi/\partial x$, $dv_{\parallel}^{(0)}/dx$ is the equilibrium flow gradient, and have chosen the equilibrium distribution function to be a Maxwellian, i.e.,

$$F_s^{(0)} \equiv n_0 \left(\frac{m_s}{2\pi T_s} \right)^{3/2} \exp\left(-\frac{B}{v_{ths}^2} \mu - \frac{1}{2} \frac{v_{\parallel}^2}{v_{ths}^2} \right).$$

For simplicity we will assume the $\mathbf{E} \times \mathbf{B}$ shear profile to have the form $v_E^{(0)}(x) = v_E^{(0)}(0) + x \partial v_E^{(0)}/\partial x$, such that the Doppler shifted frequency can be written as $\bar{\omega}_k \equiv \omega_k - v_E^{(0)}(0) k_y$. The nonresonant component of the velocity integral given by Eq. (19) can be estimated by expanding the response function given by Eq. (20) in the parameter $(v_{thi} k_{\parallel} + k_y x \partial v_E^{(0)}/\partial x) / \bar{\omega}_k < 1$, yielding

$$\left(\bar{\omega}_k - v_{\parallel} k_{\parallel} - \frac{\partial v_E^{(0)}}{\partial x} k_y x \right)^{-1} \approx \frac{1}{\bar{\omega}_k} \left\{ 1 + \frac{v_{\parallel} k_{\parallel}}{\bar{\omega}_k} + \frac{k_y x \partial v_E^{(0)}}{\bar{\omega}_k \partial x} + \frac{1}{\bar{\omega}_k^2} \left(v_{\parallel} k_{\parallel} + k_y x \frac{\partial v_E^{(0)}}{\partial x} \right)^2 + \cdots \right\}. \quad (21)$$

Substituting Eq. (20) into Eq. (19) and utilizing the approximate response function given by Eq. (21) yield an expression for the (nonresonant) fluid component of the momentum flux

$$\begin{aligned} \Pi_{\parallel}^{EB} = & in_0 m_i c_s^2 \left\langle \sum_{m,n} \frac{\omega_{ci}}{\bar{\omega}_k} (k_y \rho_s)^2 \left| \frac{e \delta \phi_{m,n}(x)}{T_e} \right|^2 \right. \\ & \times \left. \left\{ \frac{k_{\parallel}}{k_y} \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) - \frac{1}{\omega_{ci}} \frac{\partial v_{\parallel}^{(0)}}{\partial x} \right\} \right\rangle_x \\ & + in_0 m_i c_s^2 \left\langle \sum_{m,n} \frac{\omega_{ci}^2}{\bar{\omega}_k^2} (k_y \rho_s)^2 k_{\parallel} x \frac{1}{\omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x} \left(1 - 2 \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) \right. \\ & \times \left. \left| \frac{e \delta \phi_{m,n}(x)}{T_e} \right|^2 \right\rangle_x, \quad (22) \end{aligned}$$

where $\omega_e^* \equiv k_y v_e^* = k_y c_s (\rho_s/L_n)$, $\omega_{pi}^* \equiv -(1 + \eta_i) \omega_e^*/\tau$, $L_n^{-1} \equiv -d \ln n_0/dx$, $L_{Ti}^{-1} \equiv -d \ln T_i/dx$, $\eta_i \equiv L_n/L_{Ti}$, $\tau \equiv T_e/T_i$, $c_s \equiv \sqrt{T_e/m_i}$, $\rho_s \equiv c_s/\omega_{ci}$, and $\omega_{ci} \equiv eB/(m_i c)$. We have neglected finite Larmor radius corrections as well as terms quadratic in equilibrium velocity gradients for simplicity, and it is understood that only the real piece of Π_{\parallel}^{EB} is kept. The first term in Eq. (22) possesses both a residual stress term (i.e., a component of the stress not proportional to the equilibrium parallel velocity or its gradient) and a diffusive term. Near the mode rational surface, k_{\parallel} may be approximated as $k_{\parallel} \approx -(m/r)x/L_s = -k_y x/L_s$, where $L_s^{-1} \equiv \text{sgn}(B_{\theta})(r/R)(1/|q|) \times (q'/q)$, such that this term vanishes for even $|\delta \phi_{m,n}(x)|^2$. We also note that since L_s can be seen to have odd parity in B_{θ} , Eq. (22) has odd parity in the plasma current.

In order to compute the relative signs of the two terms in Eq. (22) it is useful to explicitly take the real piece, i.e., for the first term (only writing the residual stress component)

$$\begin{aligned} \left\langle \frac{\omega_k}{k_{\parallel}} \right\rangle = & -n_0 m_i c_s^2 \sum_{m,n} \frac{\omega_{ci} \Delta \omega}{\bar{\omega}_k^2} (k_y \rho_s)^2 \frac{A_s}{\omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x} \left(1 - 2 \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) \\ & \times \left\langle \left| \frac{e \delta \phi_{m,n}(x)}{T_e} \right|^2 \right\rangle_x, \quad (23) \end{aligned}$$

where $\Delta \omega$ is the turbulent decorrelation rate which we have assumed to satisfy $\bar{\omega}_k > \Delta \omega$. For simplicity we will assume the radial electric field to be dominated by its diamagnetic component, i.e., $\partial v_E^{(0)}/\partial x \approx v_{thi} \rho_i (1 + \eta_i)/L_n^2 > 0$, where we note that an inversion in the sign of the radial electric field shear would simply lead to a flip in the overall sign of Eq. (22). The only components of Eq. (23) which are not positive definite are A_s and $(1 - 2\omega_{pi}^*/\bar{\omega}_k)$. For ion temperature gradient (ITG) driven turbulence we assume $0 > \bar{\omega}_k > \omega_{pi}^*$, hence $1 - 2\omega_{pi}^*/\bar{\omega}_k < 0$, and the reduced model in Appendix B yields $A_s > 0$, such that $\langle \omega_k/k_{\parallel} \rangle > 0$. For drift wave turbulence, $1 - 2\omega_{pi}^*/\bar{\omega}_k > 0$ and $A_s < 0$, such that we again have $\langle \omega_k/k_{\parallel} \rangle > 0$. Hence the sign of this term can be seen to be independent of the underlying branch of microturbulence excited for the simple model employed. The sign of the second term may be computed directly by taking the real piece of the second term of Eq. (22), i.e.,

$$\left\langle \frac{\omega_E}{k_{\parallel}} \right\rangle = -2n_0 m_i c_s^2 \sum_{m,n} \frac{\omega_{ci}^2 \Delta \omega}{\bar{\omega}_k^3} (k_y \rho_s)^2 \frac{1}{\omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x} \times \left(1 - 3 \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) \frac{k_y}{L_s} \left\langle x^2 \left| \frac{e \delta \phi_{m,n}(x)}{T_e} \right|^2 \right\rangle_x. \quad (24)$$

For ITG turbulence, $k_y/\bar{\omega}_k < 0$ and $1 - 3\omega_{pi}^*/\bar{\omega}_k < 0$, which results in the sign being negative. Considering drift wave turbulence, $k_y/\bar{\omega}_k > 0$ and $1 - 3\omega_{pi}^*/\bar{\omega}_k > 0$, such that the sign of this term is again negative.

B. Polarization drift

In Sec. III A, a simple expression was derived for the nondiffusive residual stress term induced by $\mathbf{E} \times \mathbf{B}$ convection. While a naive ordering of the strength of this term versus that arising from the polarization drift would suggest that $\mathbf{E} \times \mathbf{B}$ convection is always dominant (i.e., $\mathbf{v}_p/\mathbf{v}_E \sim \omega_k/\omega_{ci}$), the nondiffusive residual stress terms were observed to be diminished by factors of $\langle x \rangle/L_s$ and $\omega_{ci}^{-1} \partial v_E^{(0)}/\partial x$, respectively. Furthermore, in Appendix B we show that $\langle x \rangle/L_s \sim \omega_{ci}^{-1} \partial v_E^{(0)}/\partial x$, so in regimes of weak radial electric field shear, the residual stress terms described above are significantly reduced.

The radial component of the first term in $f_{\parallel}^{(4)}$ may be rewritten as

$$\frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \langle n_0 \delta E_x \delta E_{\parallel} \rangle_x = i \frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \sum_{m,n} \left\langle n_0 \frac{\partial \delta \phi_{-m,-n}(x)}{\partial x} k_{\parallel} \delta \phi_{m,n}(x) \right\rangle_x. \quad (25)$$

Similarly, the second term in Eq. (18) can be written in the form

$$- \frac{i}{2} \frac{\partial}{\partial x} \left\langle e \sum_{m,n} k_{\parallel} \delta \phi_{-m,-n}(x) \int d^3 \bar{v} \rho_{\perp}^2 \frac{\partial \delta F_{m,n}^{(i)}(x)}{\partial x} \right\rangle_x, \quad (26)$$

where we have taken $m_e \rightarrow 0$ such that the electron contribution vanishes and $\delta F_{m,n}^{(i)}$ is given by the approximate expression in Eq. (20). It is instructive to consider Eq. (26) in two limits. First considering the limit $v_{thi} k_{\parallel} / \bar{\omega}_k > 1$, the response function given by Eq. (20) can be approximated as

$$\left(\bar{\omega}_k - v_{\parallel} k_{\parallel} - \frac{\partial v_E^{(0)}}{\partial x} k_y x \right)^{-1} \approx - \frac{1}{v_{\parallel} k_{\parallel}} \left(1 + \frac{\bar{\omega}_k}{v_{\parallel} k_{\parallel}} - \frac{k_y x}{k_{\parallel} v_{\parallel}} \frac{\partial v_E^{(0)}}{\partial x} + \dots \right). \quad (27)$$

Utilizing Eqs. (27) and (20), Eq. (26) can be approximated, yielding

$$- i \frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \left\langle n_0 \sum_{m,n} k_{\parallel} \delta \phi_{m,n}(x) \frac{\partial \delta \phi_{-m,-n}(x)}{\partial x} \right\rangle_x, \quad (28)$$

where we note that the radial derivative inside the spatial integration is a fast derivative, such that it commutes with equilibrium quantities. Also, we have made the approximation $J_0(\lambda) \approx 1$, and only the lowest order term in the expansion given by Eq. (27) has been utilized. After summing Eqs. (25) and (28), it is clear that $f_{\parallel}^{(4)}$ vanishes. Hence, in the quasistatic limit, the polarization drift can be seen to have negligible impact on the momentum budget.

Turning now to the limit of $v_{thi} k_{\parallel} / \bar{\omega}_k < 1$ (typically well satisfied for drift wave turbulence), the response function may be approximated by Eq. (20), which allows the second term in Eq. (18) to be estimated by

$$i \frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \left[n_0 \sum_{m,n} \frac{\omega_{pi}^*}{\bar{\omega}_k} \left\langle k_{\parallel} \delta \phi_{-m,-n}(x) \frac{\partial \delta \phi_{m,n}(x)}{\partial x} \right\rangle_x \right]. \quad (29)$$

Summing Eqs. (25) and (29), $f_{\parallel}^{(4)}$ can be written as

$$f_{\parallel}^{(4)} = i \frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \left[n_0 \sum_{m,n} \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) \times \left\langle \frac{\partial \delta \phi_{-m,-n}(x)}{\partial x} k_{\parallel} \delta \phi_{m,n}(x) \right\rangle_x \right]. \quad (30)$$

Equation (30) provides a transparent expression for the gyrokinetic analog of the electrostatic Maxwell stress tensor. Before proceeding further, it is useful to comment on the mathematical structure of Eq. (30). As discussed above, the gyrokinetic Poisson equation [Eq. (7)] can be seen to have a form approximately analogous to that of a dielectric medium with a permittivity given by ϵ_{\perp} . Thus, it is not surprising that $f_{\parallel}^{(4)}$ is proportional to $\nabla_{\perp} \cdot \langle n_0 \delta \mathbf{E}_{\perp} \delta E_{\parallel} \rangle$, where the factor n_0 appearing inside the divergence results from the form of the gyrokinetic Poisson equation, and the additional pressure gradient driven term in Eq. (30) emerges from finite Larmor radius corrections. Furthermore, Eq. (30) has even parity in x about the rational surface and will thus be nonvanishing even in the absence of radial electric field shear.

In order to derive an explicit expression for $f_{\parallel}^{(4)}$ it is useful to utilize a simple expression for the radial eigenmodes of the underlying turbulence, i.e.,

$$\delta \phi_{m,n}(x) = \sum_l a_l H_l(\sqrt{l} \mu_k x) \exp\left(-\frac{i}{2} \mu_k x^2\right), \quad (31)$$

where outgoing wave boundary conditions have been utilized to select the sign of the effective radial wave number $k_r(x) = \mu_k x$. Substituting Eq. (31) into Eq. (30) yields

$$\begin{aligned}
f_{\parallel}^{(4)} = & -\frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \left\{ n_0 \sum_{m,n} \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) k_y \frac{\text{Re}[\mu_{-k}]}{L_s} \langle x^2 | \delta\phi_{m,n}(x) |^2 \rangle_x \right\} - \frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \\
& \times \left\langle n_0 \sum_{m,n} \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) k_y \frac{1}{L_s} \exp(-|\text{Im}[\mu_k]|x^2) \times \text{Re} \left[i \sum_{l \neq l'} a_{l,l'} H_{l'}(\sqrt{i\mu_k x}) H_l(\sqrt{i\mu_{-k} x}) \right] \right\rangle_x \\
& - \left\langle \frac{\epsilon_{\perp}}{2\pi n_0} \frac{\partial}{\partial x} n_0 \sum_{m,n} \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) k_y \frac{1}{L_s} \exp(-|\text{Im}[\mu_k]|x^2) \times \text{Re} \left[i \sum_{l' \neq l-2} a_{l,l'} l(l-1) H_{l'}(\sqrt{i\mu_k x}) H_{l-2}(\sqrt{i\mu_{-k} x}) \right] \right\rangle_x, \quad (32)
\end{aligned}$$

where $a_{l,l'} \equiv a_l a_{l'}$, H_l is a Hermite polynomial, $l=0,1,2,\dots$, and we have used⁴⁰

$$x \frac{\partial}{\partial x} H_l(\sqrt{i\mu_k x}) = l H_l(\sqrt{i\mu_k x}) + 2l(l-1) H_{l-2}(\sqrt{i\mu_k x}).$$

A number of observations are immediately apparent. First, this result clearly has the form of a residual stress and is hence distinct from both pinch and diffusion terms. Perhaps of greater importance is that $f_{\parallel}^{(4)}$ is nonvanishing even for regimes of zero radial electric field shear. Similarly, outgoing wave boundary conditions can be seen to determine the phase between δE_{\parallel} and δE_x , and hence the sign of Eq. (32). Also, while the eigenmodes utilized in computing Eq. (32) do not contain a radial shift in the mode off the rational surface induced by radial electric field shear, this shift can instead be accounted for via the excitation of odd parity Hermite polynomials. Indeed, the presence of radial electric field shear has been shown to couple different radial mode numbers.^{41,42} Thus, in regimes of strong radial electric field shear, one might anticipate more robust contributions from the second and third terms in Eq. (32) via the excitation of higher l modes. Finally, in the limit of a purely growing mode (i.e., $\text{Re}[\mu_k] \rightarrow 0$), the first term in Eq. (32) can be immediately seen to vanish, whereas the second and third terms vanish due to the orthogonality of the Hermite polynomials.

If we now consider the idealized limit in which the $l=0$ mode is dominant (usually the most weakly damped for drift waves⁴³), the second and third terms in Eq. (32) may be neglected, yielding the simplified expression

$$\begin{aligned}
f_{\parallel}^{(4)} \approx & -\frac{\epsilon_{\perp}}{4\pi n_0} \frac{\partial}{\partial x} \\
& \times \left[n_0 \sum_{m,n} \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right) k_y \frac{\text{Re}[\mu_{-k}]}{L_s} \langle x^2 | \delta\phi_{m,n}(x) |^2 \rangle_x \right], \quad (33)
\end{aligned}$$

and we note that $\text{Re}[\mu_{-k}] = -\text{Re}[\mu_k]$. The sign of Eq. (33) can be determined by noting that $\rho_i^4 k_y \text{Re}[\mu_{-k}]/L_s = -(k_y v_{thi}/\bar{\omega}_k) |k_y| \rho_i (\rho_i/L_s)^2$, where for ITG turbulence we have $1 - \omega_{pi}^*/\bar{\omega}_k < 0$ and $k_y/\bar{\omega}_k < 0$, such that $f_{\parallel}^{(4)} > 0$. Similarly, for drift wave turbulence $1 - \omega_{pi}^*/\bar{\omega}_k > 0$ and $k_y/\bar{\omega}_k > 0$,

which leads to $f_{\parallel}^{(4)} > 0$. Finally, we note that since $f_{\parallel}^{(4)}$ appears on the RHS of the momentum equation [Eq. (4)], an additional minus sign will be introduced, such that $f_{\parallel}^{(4)}$ will often compete with the first term in Π_{\parallel}^{EB} . Note that the excitation of higher l modes can potentially alter this picture. Inclusion of these terms would require a derivation of the $a_{l,l'}$ coefficients, which is beyond the scope of the present work.

Comparing the magnitude of Eq. (33) with Eq. (22) yields for the first term

$$\frac{f_{\parallel}^{(4)}}{\langle \omega_k/k_{\parallel} \rangle} \sim \frac{1}{A_s} \frac{\omega_e^* \omega_e^* \bar{\omega}_k}{\omega_{ci} v_E' \Delta\omega} \frac{1}{(k_y \rho_s)^2} \left(\frac{L_n}{L_s} \right)^2 \frac{1}{\rho_s^2} \frac{\langle x^2 | \delta\phi_{m,n}(x) |^2 \rangle_x}{\langle | \delta\phi_{m,n}(x) |^2 \rangle_x}. \quad (34)$$

If we assume that the radial electric field is dominated by the diamagnetic term, i.e., $v_E' \approx v_{thi} \rho_i (1 + \eta_i)/L_n^2$, the radial extent of the mode is set by the points of strong ion Landau damping (i.e., $v_{thi} k_{\parallel} = \bar{\omega}_k$), and we approximate A_s by its value for drift wave turbulence (i.e., $A_s \approx L_s/L_n$). Equation (34) can be rewritten as

$$\frac{f_{\parallel}^{(4)}}{\langle \omega_k/k_{\parallel} \rangle} \sim \frac{L_n}{L_s} \left(\frac{\tau}{1 + \eta_i} \right) \left(\frac{\bar{\omega}_k}{\omega_e^*} \right)^2 \frac{\bar{\omega}_k}{\Delta\omega}. \quad (35)$$

Some comments on these orderings are appropriate at this point. While L_n/L_s is typically small, $\bar{\omega}_k/\Delta\omega$ is typically greater than one, such that neither of these terms should be considered negligible *a priori*. Furthermore, the presence of a factor of τ^2 suggests sensitivity to the heating mechanism utilized. Also, while an estimation of the radial electric field based on the diamagnetic term may at times provide a reasonable approximation, contributions from the poloidal term may significantly alter the above ordering.

While our primary focus up to this point has been the comparison of different turbulent contributions to the momentum flux, here it is useful to compare with results from neoclassical theory. As an instructive example we compare the nondiffusive neoclassical term discussed in Ref. 44 (derived from the high collisionality regime) with the turbulent

contribution derived in this analysis. The momentum flux induced by the former term can be written as

$$\Pi_{\text{neo}} \approx \frac{0.11 \eta q^2 B_\phi}{1 + Q^2/S^2 B_\theta} U_\theta \frac{\partial \ln T_i}{\partial r}, \quad (36a)$$

where $\eta \approx 1.2 n_i m_i \rho_i^2 \nu_i$ and

$$\frac{Q^2}{S^2} \equiv 0.51 \left(\frac{qR\nu_i}{v_{thi}} \right) \frac{B_\phi \rho_i}{B_\theta L_{Ti}} \times \left[\left(v_{thi} \rho_i \frac{\partial \ln T_i}{\partial r} \right)^{-1} U_\theta - 0.625(1 + 2\eta_i^{-1}) \right]. \quad (36b)$$

We take the poloidal flow to be neoclassical,⁴⁵ i.e.,

$$U_\theta \approx -1.83 v_{thi} \rho_i \frac{\partial \ln T_i}{\partial r}. \quad (37)$$

In order to compare the magnitude of the turbulent momentum flux derived above, we estimate the magnitude of the turbulent fluctuations by their mixing length level, i.e., $|e \delta \phi / T_e|^2 \approx \rho_s^2 / L_n^2$. Utilizing this mixing length estimate, as well as estimations analogous to those utilized above, the ratio of Eq. (36) with the momentum flux induced by the polarization drift [see Eq. (33)] can be estimated as

$$\frac{\Pi_{\text{neo}}}{\Pi_{\text{pol}}} \sim \frac{0.24 \varepsilon^{3/2} \nu^* q^2 \eta_i^2 \tau^{-5/2} L_n}{1 + \eta_i \frac{k_y \rho_s}{r}}, \quad (38)$$

where $\nu^* \equiv \varepsilon^{-3/2} \nu_i q R / v_{thi}$, $\varepsilon \equiv r/R$, and for simplicity we have taken $Q^2/S^2 \leq 1$. While this estimate is clearly very primitive, it is apparent that for sufficiently high collisionality the neoclassical contribution is dominant. However, in the limit of low collisionality it appears likely that the polarization drift contribution to the momentum flux, as well as its $\mathbf{E} \times \mathbf{B}$ shear driven counterpart, is likely dominant in a wide range of parameter regimes (see Refs. 46 and 47 for neoclassical contributions to the momentum flux in low collisionality regimes).

IV. DISCUSSION AND CONCLUSION

In this paper, a novel nondiffusive contribution to the radial flux of parallel momentum has been derived. This contribution, which arises from the parallel nonlinearity within the gyrokinetic framework, appears due to the polarization drift. More specifically, the polarization drift is manifested in gyrokinetics by a violation of gyrocenter quasineutrality. As discussed above, the gyrokinetic Poisson equation has an analogous form to the Poisson equation for a dielectric medium, and hence this mechanism can be understood to correspond to the generalization of the electrostatic Maxwell stress tensor to the gyrokinetic framework. Furthermore, this mechanism does not require mean radial electric field shear, and is thus likely to be active in a wide range of plasma

regimes. We emphasize that while this contribution to the momentum flux is nominally higher order in an expansion in $\varepsilon \sim \rho_i / L_n$, a detailed analysis demonstrates that this term can be comparable to its mean $\mathbf{E} \times \mathbf{B}$ shear driven counterpart. Furthermore, we note that while this term has a mathematical form, which is similar to the mean $\mathbf{E} \times \mathbf{B}$ shear component of the momentum flux, it has a physically distinct origin. While a quantitative calculation of the net residual stress term is beyond the scope of the present analysis, the addition of a novel contribution to the momentum flux provides a new candidate for understanding offsets in plasma rotation, which have been observed in a wide range of plasma devices. The role of this polarization induced residual stress in the developing theory of intrinsic rotation will be discussed in a future publication.

ACKNOWLEDGMENTS

We would like to thank F. Hinton for stimulating discussions on neoclassical transport theory.

This research was supported by the U.S. Department of Energy (Contract Nos. DE-FG02-04ER54738, DE-FC02-08ER54959, and DE-FC02-08ER54983).

APPENDIX A: DERIVATION OF $f_{\parallel}^{(4)}$

Here we provide a detailed derivation of the form of $f_{\parallel}^{(4)}$ analyzed in Sec. III. The third term in Eq. (15) can be easily seen to vanish since it is linear in a fluctuating quantity. Considering the second term in Eq. (15), after utilizing Eq. (11) this term can be written as

$$\begin{aligned} & \left\langle \sum_s m_s \int d^3 \bar{v} \dot{V}_{\parallel}^{(3)} \delta F_s^{(1)} \right\rangle \\ &= - \left\langle \sum_s q_s \int d^3 \bar{v} \delta F_s^{(1)} \hat{b} \cdot \nabla J_0^{(0)} \delta \phi^{(2)} \right\rangle \\ & \quad - \left\langle \sum_s q_s \int d^3 \bar{v} \delta F_s^{(1)} \hat{b} \cdot \nabla J_0^{(1)} \delta \phi^{(1)} \right\rangle. \end{aligned} \quad (A1)$$

Utilizing Eq. (12), Eq. (A1) can be rewritten as

$$\begin{aligned} & \left\langle \sum_s m_s \int d^3 \bar{v} \dot{V}_{\parallel}^{(3)} \delta F_s^{(1)} \right\rangle \\ &= \frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(0)})^2 \delta \phi^{(1)} \hat{b} \cdot \nabla \delta \phi^{(2)} \rangle \\ & \quad - \left\langle \sum_s q_s \int d^3 \bar{v} \delta F_s^{(1)} \hat{b} \cdot \nabla J_0^{(1)} \delta \phi^{(1)} \right\rangle. \end{aligned} \quad (A2)$$

Similarly, the first term in Eq. (15) after utilizing Eqs. (9) and (17) can be written as

$$\begin{aligned}
\left\langle \sum_s m_s \int d^3\bar{v} V_{\parallel}^{(2)} \delta F_s^{(2)} \right\rangle &= \frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(0)})^2 \delta\phi^{(2)} \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle + \frac{\epsilon_{\perp}}{4\pi} \langle \nabla_{\perp}^{(0)} \cdot \nabla_{\perp}^{(1)} \delta\phi^{(1)} \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle \\
&+ \frac{\epsilon_{\perp}}{4\pi n_0} \langle \nabla_{\perp}^{(0)} \cdot (\delta n_i^{(1)} \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle + \frac{\epsilon_{\perp}}{4\pi n_0} \langle \nabla_{\perp}^{(1)} \cdot (n_0 \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle \\
&+ \left\langle \sum_s q_s \int d^3\bar{v} J_0^{(1)} \delta F_s^{(1)} \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle. \tag{A3}
\end{aligned}$$

Summing Eqs. (A3) and (A2), $f_{\parallel}^{(4)}$ can be written in the form

$$\begin{aligned}
f_{\parallel}^{(4)} &= \frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(0)})^2 \delta\phi^{(1)} \hat{b} \cdot \nabla \delta\phi^{(2)} \rangle + \frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(0)})^2 \delta\phi^{(2)} \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle + \frac{\epsilon_{\perp}}{4\pi} \langle \nabla_{\perp}^{(0)} \cdot (\nabla_{\perp}^{(1)} \delta\phi^{(1)}) \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle \\
&+ \frac{\epsilon_{\perp}}{4\pi n_0} \langle \nabla_{\perp}^{(1)} \cdot (n_0 \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle + \left\langle \sum_s q_s \int d^3\bar{v} J_0^{(1)} \delta F_s^{(1)} \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle - \left\langle \sum_s q_s \int d^3\bar{v} \delta F_s^{(1)} \hat{b} \cdot \nabla J_0^{(1)} \delta\phi^{(1)} \right\rangle \\
&+ \frac{\epsilon_{\perp}}{4\pi n_0} \langle \nabla_{\perp}^{(0)} \cdot (\delta n_i^{(1)} \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle. \tag{A4}
\end{aligned}$$

Equation (A4) can be greatly simplified. Considering the first two terms on the RHS of Eq. (A4), after multiple integrations by parts of the first term, these terms can be written as

$$1\text{st} + 2\text{nd} = -\frac{\epsilon_{\perp}}{4\pi} \langle \hat{b} \cdot \nabla \delta\phi^{(1)} (\nabla_{\perp}^{(0)})^2 \delta\phi^{(2)} \rangle + \frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(0)})^2 \delta\phi^{(2)} \hat{b} \cdot \nabla \delta\phi^{(1)} \rangle = 0. \tag{A5}$$

Considering the third and fourth term on the RHS of Eq. (A4), after applying the product rule, and integrating by parts, these terms can be written as

$$\begin{aligned}
3\text{rd} + 4\text{th} &= -\frac{\epsilon_{\perp}}{4\pi} \langle (\nabla_{\perp}^{(1)} \delta\phi^{(1)}) \cdot (\hat{b} \cdot \nabla) \nabla_{\perp}^{(0)} \delta\phi^{(1)} \rangle - \frac{\epsilon_{\perp}}{4\pi} \langle \nabla_{\perp}^{(1)} \cdot (\hat{b} \cdot \nabla \nabla_{\perp}^{(0)} \delta\phi^{(1)}) \delta\phi^{(1)} \rangle + \frac{\epsilon_{\perp}}{4\pi n_0} \langle \nabla_{\perp}^{(1)} n_0 \cdot \langle \delta \mathbf{E}_{\perp}^{(1)} \delta E_{\parallel}^{(2)} \rangle \rangle \\
&= -\frac{\epsilon_{\perp}}{4\pi} \nabla_{\perp}^{(1)} \cdot \langle \delta\phi^{(1)} \hat{b} \cdot \nabla (\nabla_{\perp}^{(0)} \delta\phi^{(1)}) \rangle + \frac{\epsilon_{\perp}}{4\pi n_0} \langle \nabla_{\perp}^{(1)} n_0 \cdot \langle \delta \mathbf{E}_{\perp}^{(1)} \delta E_{\parallel}^{(2)} \rangle \rangle \\
&= \frac{\epsilon_{\perp}}{4\pi n_0} \nabla_{\perp}^{(1)} \cdot \langle n_0 \delta \mathbf{E}_{\perp}^{(1)} \delta E_{\parallel}^{(2)} \rangle. \tag{A6}
\end{aligned}$$

Similarly, the fifth and sixth terms in Eq. (A4) can be simplified, i.e.,

$$\begin{aligned}
5\text{th} + 6\text{th} &= -\frac{1}{4} \left\langle \sum_s q_s \int d^3\bar{v} \rho_{\perp}^2 \delta F_s^{(1)} \hat{b} \cdot \nabla (\nabla_{\perp}^{(0)} \cdot \nabla_{\perp}^{(1)} + \nabla_{\perp}^{(1)} \cdot \nabla_{\perp}^{(0)}) \delta\phi^{(1)} \right\rangle \\
&+ \frac{1}{4} \left\langle \sum_s q_s \int d^3\bar{v} \rho_{\perp}^2 (\nabla_{\perp}^{(0)} \cdot \nabla_{\perp}^{(1)} + \nabla_{\perp}^{(1)} \cdot \nabla_{\perp}^{(0)}) \delta F_s^{(1)} \hat{b} \cdot \nabla \delta\phi^{(1)} \right\rangle,
\end{aligned}$$

after rearranging terms, and integrating by parts this expression can be simplified as

$$5\text{th} + 6\text{th} = -\frac{1}{2} \nabla_{\perp}^{(1)} \cdot \left\langle \sum_s q_s \int d^3\bar{v} \rho_{\perp}^2 (\nabla_{\perp}^{(0)} \delta F_s^{(1)}) \delta E_{\parallel}^{(2)} \right\rangle. \tag{A7}$$

Considering the last term in Eq. (A4), after integrating by parts once this term can be written as

$$\begin{aligned}
7\text{th} &= -\frac{\epsilon_{\perp}}{4\pi n_0} \langle \delta n_i^{(1)} (\nabla_{\perp}^{(0)} \delta\phi^{(1)}) \cdot \hat{b} \cdot \nabla (\nabla_{\perp}^{(0)} \delta\phi^{(1)}) \rangle \\
&= -\frac{\epsilon_{\perp}}{8\pi n_0} \langle \delta n_i^{(1)} \hat{b} \cdot \nabla |\nabla_{\perp}^{(0)} \delta\phi^{(1)}|^2 \rangle = 0. \tag{A8}
\end{aligned}$$

Summing Eqs. (A6) and (A7) yields Eq. (18).

APPENDIX B: RADIAL EIGENMODE ANALYSIS

In this appendix we seek to derive simple expressions for the radial eigenmodes of the underlying electrostatic microturbulence in the presence of radial electric field shear.

For simplicity we will utilize a simplified geometry consistent with that utilized in the main body of the text and limit ourselves to the fluid limit, i.e., $v_{thi}|k_{\parallel}|/\bar{\omega}_k < 1$. From the gyrokinetic Poisson equation [Eq. (12)] and the linear response given by Eq. (20) an eigenmode equation for the electrostatic microturbulence may be approximated as

$$-\epsilon_{\perp}(k)k_{\perp}^2\delta\phi_k = \epsilon_{\perp}(k)k_{\perp}^2\chi_k\delta\phi_k, \quad (\text{B1a})$$

where

$$\begin{aligned} \epsilon_{\perp}(k)k_{\perp}^2\chi_k \approx & k_{\text{De}}^2 \left[1 - \frac{\omega_e^*}{\omega_k} \Gamma_0(b) \left(1 + \frac{\eta_i}{2} - \frac{\eta_i}{\eta_c} \right) \right] \\ & - k_{\text{De}}^2 \left(\frac{c_s k_{\parallel}}{\omega_k} \right)^2 \left(1 - \frac{\omega_{pi}^*}{\omega_k} - \frac{k_y}{k_{\parallel}} \frac{1}{\omega_{ci}} \frac{\partial v_{\parallel}^{(0)}}{\partial x} \right), \end{aligned} \quad (\text{B1b})$$

where $\eta_c^{-1} \equiv (1/2) + b[1 - I_1(b)/I_0(b)]$, $b \equiv -\rho_i^2 \partial^2 / \partial x^2 + k_y^2 \rho_i^2$, $\Gamma_0(b) \equiv I_0(b) \exp(-b)$, I_n is the n th order modified Bessel function, and we have neglected contributions from mean velocity gradients and finite Larmor radius corrections in the second term in Eq. (B1). Accounting for $\mathbf{E} \times \mathbf{B}$ shear via introducing the Doppler shift $\omega_k \rightarrow \bar{\omega}_k - k_y x \partial v_E^{(0)} / \partial x$, Eq. (B1) can be rewritten as¹⁸

$$0 = \rho_s^2 \frac{\partial^2}{\partial x^2} \delta\phi_{m,n}(x) + Q(x) \delta\phi_{m,n}(x), \quad (\text{B2a})$$

$$\begin{aligned} Q(x) \equiv & -k_y^2 \rho_s^2 - \alpha^{-1} \left(1 - \frac{\omega_e^*}{\bar{\omega}_k} \right) + \frac{1}{\alpha} \frac{\omega_e^*}{\bar{\omega}_k^2} \\ & \times \left(\sigma \frac{\partial v_E^{(0)}}{\partial x} + \frac{L_n}{L_s} \frac{\partial v_{\parallel}^{(0)}}{\partial x} \right) k_y x + \left(\frac{c_s k_y}{\bar{\omega}_k} \right)^2 \frac{x^2}{L_s^2}, \end{aligned} \quad (\text{B2b})$$

where we have defined

$$\begin{aligned} \alpha & \equiv \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right), \\ \sigma & \equiv 1 + \left(\frac{1 + \eta_i}{\tau} \right) \left(1 - \frac{\omega_{pi}^*}{\bar{\omega}_k} \right)^{-1} \left(1 - \frac{\omega_e^*}{\bar{\omega}_k} \right). \end{aligned}$$

Equation (B2) can be rewritten in the form

$$Q(x) \equiv -k_y^2 \rho_s^2 - \alpha^{-1} \left(1 - \frac{\omega_e^*}{\bar{\omega}_k} \right) + \left(\frac{c_s k_y}{\bar{\omega}_k} \right)^2 \left(\frac{x - x_0}{L_s} \right)^2, \quad (\text{B3})$$

where

$$\frac{x_0}{L_s} = -\frac{1}{2} \frac{L_s}{\alpha L_n \omega_{ci}} \left(\sigma \frac{\partial v_E^{(0)}}{\partial x} + \frac{L_n}{L_s} \frac{\partial v_{\parallel}^{(0)}}{\partial x} \right), \quad (\text{B4})$$

and for simplicity we consider the limit of weak equilibrium flow gradients such that we can neglect terms proportional to x_0^2/L_s^2 . Equations (B2) and (B3) have solutions of the form

$$\delta\phi_{m,n}(x) = H_l(\sqrt{i\mu_k}(x - x_0)) \exp\left[-\frac{i}{2}\mu_k(x - x_0)^2\right], \quad (\text{B5})$$

where $\rho_i \mu_k \equiv (v_{thi}|k_y|/\bar{\omega}_k)L_s^{-1}$ (the magnitude ensures that the imaginary component of μ_k is even in k_y such that the solution is always convergent) and whose eigenvalues are determined by

$$0 = -i(2l + 1) \left(\frac{v_{thi}|k_y|}{\bar{\omega}_k} \right) \frac{\rho_i}{L_s} - k_y^2 \rho_i^2 - \alpha^{-1} \left(1 - \frac{\omega_e^*}{\bar{\omega}_k} \right). \quad (\text{B6})$$

Equation (B6) can be solved yielding the two roots given by

$$\bar{\omega}_k^{\text{ITG}} \approx i(2l + 1) \frac{L_n}{L_s} |\omega_{pi}^*| \left(1 + k_y^2 \rho_s^2 \frac{\omega_{pi}^*}{\omega_e^*} \right)^{-1}, \quad (\text{B7a})$$

$$\bar{\omega}_k^{\text{DW}} \approx \frac{\omega_e^*}{1 + k_y^2 \rho_s^2} \left(1 + k_y^2 \rho_s^2 \frac{\omega_{pi}^*}{\omega_e^*} \right), \quad (\text{B7b})$$

where the first solution corresponds to an unstable ITG branch and the second to a stable drift wave branch. While the above model is far too simple to provide a quantitative description of drift wave/ITG turbulence, it does provide a simple basis for understanding qualitative elements of these two branches of turbulence.

In order to estimate the strength of the first term in Eq. (22) for both drift wave and ITG turbulence it is necessary to derive estimates for the shift in the eigenmodes off their respective rational surfaces. Considering ITG turbulence first, σ can be estimated as

$$\begin{aligned} \sigma & \approx 1 - \left[1 - i(2l + 1) \frac{L_n}{L_s} \text{sgn}(k_y) \right] \\ & \times \left[1 - i(2l + 1) \frac{L_n}{L_s} \text{sgn}(k_y) \left(\frac{1 + \eta_i}{\tau} \right) \right] \\ & \approx i \text{sgn}(k_y) (2l + 1) \frac{L_n}{L_s} \left[1 + \left(\frac{1 + \eta_i}{\tau} \right) \right], \end{aligned} \quad (\text{B8})$$

where we note that the first and second terms in σ cancel to lowest order in L_n/L_s . Substituting Eq. (B8) into Eq. (B4), x_0 can be approximated as

$$\frac{x_0^{\text{ITG}}}{L_s} \approx \frac{1}{2} (2l + 1) \frac{L_n}{L_s} \left[1 + \left(\frac{1 + \eta_i}{\tau} \right) \right] \frac{1}{\omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x}, \quad (\text{B9})$$

where we have only kept contributions due to electric field shear. Inclusion of parallel flow shear in this expression would simply introduce a correction to the diffusion term in Eq. (22). We also note that within this simple model, symmetry breaking is more robust for higher l modes. This result is highly sensitive to the detailed form of the dispersion relationship and is thus likely an artifact of the simple model employed. Turning now to the drift wave branch, x_0 can be estimated to be

$$\frac{x_0^{\text{DW}}}{L_s} \approx -\frac{1}{2} \left[1 + \left(\frac{1 + \eta_i}{\tau} \right) \right]^{-1} \frac{L_s}{L_n \omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x}, \quad (\text{B10})$$

where the strength of the symmetry breaking is amplified by a factor of $(L_s/L_n)^2$ in comparison to Eq. (B9). From Eqs. (B9), (B10), and (B5) the integral over x in Eq. (22) can be easily evaluated as

$$\left\langle \frac{k_{\parallel}}{k_y} |\delta\phi_{m,n}(x)|^2 \right\rangle_x = -\frac{x_0}{L_s} \langle |\delta\phi_{m,n}(x)|^2 \rangle_x$$

$$= -A_s \frac{1}{\omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x} \langle |\delta\phi_{m,n}(x)|^2 \rangle_x, \quad (\text{B11})$$

where we have defined

$$\frac{x_0}{L_s} \equiv A_s \frac{1}{\omega_{ci}} \frac{\partial v_E^{(0)}}{\partial x},$$

with A_s defined by the coefficients in Eqs. (B9) and (B10) for ITG and drift waves, respectively.

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